Dynare

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November, 2021

Outline

Ramsey model

$$
\begin{aligned}
a_t &= \rho a_{t-1} + \varepsilon_t \\
y_t &= e^{a_t} k_t^{\alpha} \\
y_t &= c_t + i_t \\
k_{t+1} &= i_t + (1-\delta) k_t \\
\frac{c_{t+1}}{c_t} &= \beta \left(\alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} + 1 - \delta \right)
\end{aligned}
$$

Dynare deals with more general nonlinear dynamic models:

- backward *and* forward terms (decisions depend on history and expectations).
- risk (future is unknown).

↪ Dynare syntax

Problems to be solved

$$\mathbb{E}\left[f_\theta(y_{t-1}, y_t, y_{t+1}, \varepsilon_t)\middle|\Omega_t\right] = 0 \text{ for } t \in \mathbb{T}$$

with

- y_t a vector of endogenous variables.
- ε_t a vector of innovations.
- f_θ a continuous non linear function indexed by a vector of parameters θ.
- Ω_t an information set (filtration, *i.e.* $\Omega_t \subseteq \Omega_{t+s} \forall s \geq 0$).
- $\mathbb{E}[\bullet|\Omega_t]$ the conditional expectation operator.
- $\mathbb{T}$ a discrete time set (finite or not).

Problems to be solved

Perfect foresight models

- $\mathbb{T}$ is finite: $\{1, \dots, T\}$.
- Agents make their decisions in period 1.
- $\Omega_t = \Omega_T$ for all t (no surprises, everything is known in period 1).
- y_0 and y_{T+1} given.

$$f_\theta(y_{t-1}, y_t, y_{t+1}, \varepsilon_t) = 0 \text{ for } t \in \{1, \dots, T\}$$

$$\text{with, for instance, } \varepsilon_1 \neq 0 \text{ and } \varepsilon_t = 0 \quad \forall t > 1$$

- The solution is a path for the endogenous variables.
- Since decisions are made in $t = 1$, the impulse ϵ_1 is not expected.
- A priori a shock at time $t > 1$ is expected (though with some work it is possible to implement unexpected impulses in $t > 1$).
- Typical experience: How does the economy behave today if the government announce a VAT cut for next year? By how much the households will postpone consumption?

Problems to be solved

Stochastic models

- $\mathbb{T}$ is not finite: $\mathbb{N}$ or $\mathbb{Z}$.
- $\Omega_t = \{\underline{y_{t-1}}, \varepsilon_{\underline{t}}\}$ for all t, the parameterized model, f_θ, and the invariant distribution of the innovations, ε, are also in the information set.

$$\mathbb{E}\left[f_\theta(y_{t-1}, y_t, y_{t+1}, \varepsilon_t)\middle|\Omega_t\right] = 0$$

- The solution is an invariant mapping between y_t and (y_{t-1}, ε_t).
- Future realizations of the impulses are unknown (but the distribution is known).
- With some work, it is possible to consider smaller information sets (limited rational expectation, learning).
- Typical experience: Does economic fuctuations have a cost? If the cycle is found to be costly, which policy could dampen the fluctuations?

Problems to be solved

Statistical inference

- The parameters, θ and the variance of the innovations Σ, are supposed to be known to the agents in the economy (in RE or PF models, but not in a LRE or learning models)...
- Economists may have some informations about these parameters, but they do not know them with certainty.
- We need to estimate these parameters (Bayesian inference, or ML approach).
- Also we need to compare the fit of different models.

Problems to be solved

Other stuff

- Optimal policy.
- Identification.
- Sensitivity.

Perfect foresight models

- Suppose that $\varepsilon_t = 0$ for all $t \geq 1$.
- The initial state of the economy, y_0, is given.
- We define $y^\star$ as the steady state of the dynamical system for the endogenous variables:

$$f_\theta\left(y^\star, y^\star, y^\star, 0\right) = 0$$

- We assume that the economy reaches $y^\star$ in finite time, by imposing $y_{T+1} = y^\star$.
- We have boundary conditions and

$$\begin{aligned} f_\theta(y_0, y_1, y_2, 0) &= 0 \\ f_\theta(y_{t-1}, y_t, y_{t+1}, 0) &= 0 \text{ for all } t = 2, \dots, T-1 \\ f_\theta(y_{T-1}, y_T, y^\star, 0) &= 0 \end{aligned}$$

$\Rightarrow$ $T \times n$ unknows and $T \times n$ equations!

Example Saddle path property in the Ramsey model.

Perfect foresight models

- We need to solve a huge system of nonlinear equations.
- A Newton algorithm is used.
- Let

$$F(\mathbf{Y}) = 0$$

  be the stacked system of non linear equations.
- Set an initial guess $\mathbf{Y}^{(0)}$, usually the steady state: $y^\star \otimes \vec{e}_T$
- Update the solution paths, $\mathbf{Y}^{(i+1)}$ $(i = 0, 1, \dots)$, by solving the following linear system of equations:

$$F\left(\mathbf{Y}^{(i)}\right) + J_F\left(\mathbf{Y}^{(i)}\right)\left(\mathbf{Y}^{(i+1)} - \mathbf{Y}^{(i)}\right) = 0$$

  where $J_F(\mathbf{Y}) = \frac{\partial F(\mathbf{Y})}{\partial \mathbf{Y}'}$ is the jacobian matrix of F.
- Stop the iterations if

$$||F\left(\mathbf{Y}^{(i)}\right)|| < \epsilon$$

- Different methods are available to solve the systems of linear equations (we do not need to explicitely inverse the jacobian matrix).

Perfect foresight models

- The size of the jacobian is very large. If we have a model with $n = 100$ endogenous variables and $T = 400$, we must solve systems of 40000 linear equations!

- This jacobian matrix is sparse:

$$
J_F(\mathbf{Y}) = \begin{pmatrix}
f_y^1 & f_{y_+}^1 & 0 & \dots & \dots & \dots & \dots & 0 \\
f_{y_-}^2 & f_y^2 & f_{y_+}^2 & 0 & \dots & \dots & \dots & 0 \\
0 & f_{y_-}^3 & f_y^3 & f_{y_+}^2 & 0 & \dots & \dots & 0 \\
 & & \ddots & \ddots & \ddots & \ddots & & \\
0 & \dots & \dots & 0 & f_{y_-}^{T-2} & f_y^{T-2} & f_{y_+}^{T-2} & 0 \\
0 & \dots & \dots & \dots & 0 & f_{y_-}^{T-1} & f_y^{T-1} & f_{y_+}^{T-1} \\
0 & \dots & \dots & \dots & \dots & 0 & f_{y_-}^{T} & f_y^{T}
\end{pmatrix}
$$

with $f_y^t = \frac{\partial F(\mathbf{Y})}{\partial y_t'}$, $f_{y_+}^t = \frac{\partial F(\mathbf{Y})}{\partial y_{t+1}'}$, and $f_{y_-}^t = \frac{\partial F(\mathbf{Y})}{\partial y_{t-1}'}$

- We exploit the sparsity when solving the linear system.

Rational expectation models

General problem

- We consider the following type of model:

$$\mathbb{E}_t \left[f(y_{t+1}, y_t, y_{t-1}, u_t) \right] = 0$$

with:

$$\begin{aligned} u_t &= \sigma \varepsilon_t \\ \mathbb{E}[\varepsilon_t] &= 0 \\ \mathbb{E}[\varepsilon_t \varepsilon_t'] &= \Sigma_\epsilon \end{aligned}$$

where σ is a scale parameter, ε is a vector of auxiliary random variables.

- **Assumption** $f: \quad \mathbb{R}^{3n+q} \rightarrow \mathbb{R}^n$ is a differentiable function in $\mathcal{C}^k$.

Rational expectation models

Solution

- The solution is the unknown function g:

$$y_t = g(y_{t-1}, u_t, \sigma)$$

- Then, we have:

$$\begin{aligned} y_{t+1} &= g(y_t, u_{t+1}, \sigma) \\ &= g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma) \end{aligned}$$

- So we can define:

$$F_g(y_{t-1}, u_t, u_{t+1}, \sigma) = f(g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma), g(y_{t-1}, u_t, \sigma), y_{t-1}, u_t)$$

- And our problem can be restated as:

$$\mathbb{E}_t \left[F_g(y_{t-1}, u_t, u_{t+1}, \sigma) \right] = 0$$

- To solve the model we have to identify the unknown function g (solve a functional equation).

Rational expectation models

First order approximation

- Let $\hat{y} = y_{t-1} - \bar{y}$, $u = u_t$, $u_+ = u_{t+1}$, $f_{y_+} = \frac{\partial f}{\partial y_{t+1}}$, $f_y = \frac{\partial f}{\partial y_t}$, $f_{y_-} = \frac{\partial f}{\partial y_{t-1}}$, $f_u = \frac{\partial f}{\partial u_t}$, $g_y = \frac{\partial g}{\partial y_{t-1}}$, $g_u = \frac{\partial g}{\partial u_t}$, $g_\sigma = \frac{\partial g}{\partial \sigma}$.
Where all the derivates are evaluated at the deterministic steady state, $y^\star$.

- With a first order Taylor expansion of F around $y^\star$:

$$\begin{aligned} 0 \simeq {F_g}^{(1)}(y_-, u, u_+, \sigma) = \\ & f_{y_+}\left(g_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + g_u u_+ + g_\sigma\sigma\right) \\ & + f_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + f_{y_-}\hat{y} + f_u u \end{aligned}$$

- **What has changed?** We now have three unknown "parameters" (g_y, g_u and g_σ) instead of an infinite number of parameters (function g).

Rational expectation models

First order approximation

- Taking the expectation conditional on the information at time t, we have:

$$\begin{aligned} 0 \simeq & f_{y_+}\left(g_y\left(g_y\hat{y} + g_u u + g_\sigma \sigma\right) + g_u \mathbb{E}_t[u_+] + g_\sigma \sigma\right) \\ & + f_y\left(g_y \hat{y} + g_u u + g_\sigma \sigma\right) + f_{y_-}\hat{y} + f_u u \end{aligned}$$

- Or equivalently:

$$\begin{aligned} 0 \simeq & \left(f_{y_+} g_y g_y + f_y g_y + f_{y_-}\right)\hat{y} + \left(f_{y_+} g_y g_u + f_y g_u + f_u\right) u \\ & + \left(f_{y_+} g_y g_\sigma + f_{y_+} g_\sigma + f_y g_\sigma\right)\sigma \end{aligned}$$

- This "equality" must hold for any value of $(\hat{y}, u, \sigma)$, so that the terms between parenthesis must be zero. We have three (multivariate) equations and three (multivariate) unknowns:

$$\begin{cases} 0 & = f_{y_+} g_y g_y + f_y g_y + f_{y_-} \\ 0 & = f_{y_+} g_y g_u + f_y g_u + f_u \\ 0 & = f_{y_+} g_y g_\sigma + f_{y_+} g_\sigma + f_y g_\sigma \end{cases}$$

Rational expectation models

First order approximation

- We must have:

$$\left(f_{y_+} g_y g_y + f_y g_y + f_{y_-}\right) \hat{y} = 0 \quad \forall \hat{y}$$

- This is a quadratic equation, but the unknown is a matrix! It is generally impossible to solve this equation analytically as we would do for a univariate quadratic equation.

- If we interpret g_y as a lead operator, we can rewrite the equation as a second order recurrent equation:

$$f_{y_+} \hat{y}_{t+1} + f_y \hat{y}_t + f_{y_-} \hat{y}_{t-1} = 0$$

- For a given initial condition, $\hat{y}_{t-1}$, an infinity of paths $(\hat{y}_t, \hat{y}_{t+1})$ is solution of the second order recurrent equation.

⇝ In the phase diagram of the Ramsey model, an infinity of trajectories satisfy the Euler and transition equations.

Rational expectation models

First order approximation

⇒ When we solve the quadratic equation for g_y, we must check that this allows to pin down a unique path for $(\hat{y}_t, \hat{y}_{t+1})$ leading to the steady state in the long run.

- To solve this quadratic equation we use a (real) generalized Schur decomposition (available in Lapack).

- In the end we obtain the following reduced form:

$$y_t = y^\star(\theta) + g_y(\theta)\left(y_{t-1} - y^\star\right) + g_u(\theta)\varepsilon_t$$

- The reduced form parameters are highly non linear function of the structural parameters θ. In general we do not have closed form expressions for $g_y(\theta)$ and $g_u(\theta)$.

Rational expectation models

Higher order approximation

- With a first order approximation, risk does not matter (certainty equivalence).

- Dynare (4.6) can compute the solutions of arbitrary order approximations of the original structural model.

Bayesian inference

- Suppose that the economist has some beliefs about the parameters, and that these beliefs can be represented by a probability density function $p(\theta)$.
- Suppose that the economist observe some variables. Let $\mathcal{Z}_T = \{z_1, z_2, \dots, z_T\}$ be the sample where z_t is a subset of y_t.
- Is it possible to update the economist's beliefs with the data? How?

Bayesian inference

- Any stochastic model defines implicitely a distribution for the endogenous variables.

- For instance, the reduced form model:

$$y_t = y^\star(\theta) + g_y(\theta)\left(y_{t-1} - y^\star\right) + g_u(\theta)\varepsilon_t$$

  implies that y_t conditional on y_{t-1} is Gaussian:

$$y_t|y_{t-1} \sim \mathcal{N}\left(\left(I_n - g_y(\theta)\right)y^\star(\theta) + g_y(\theta)y_{t-1}, g_u(\theta)\Sigma g_u(\theta)'\right)$$

  provided that $\varepsilon_t \sim \mathcal{N}\left(0, \Sigma\right)$.

- Obviously this implies that $(y_1, \dots, y_T)$ conditional on y_0 is also Gaussian. If y_0 is distributed as y_∞, then $(y_1, \dots, y_T)$ is (marginally) Gaussian:

$$(y_1, \dots, y_T)' \sim \mathcal{N}\left(y^\star(\theta) \otimes e_T, \Omega_y(\theta)\right)$$

Bayesian inference

$\Rightarrow$ Under the same assumptions the sample, $\mathcal{Z}_T$, is also Gaussian.

$$(z_1, \dots, z_T)' \sim \mathcal{N}\left(z^\star(\theta) \otimes e_T, \Omega_z(\theta)\right)$$

where $z^\star = Sy^\star$ and $\Omega_z(\theta) = S\Omega_z(\theta)S'$, with S a selection matrix.

- The density of the sample is also called the *likelihood*:

$$p(\mathcal{Z}_T|\theta) = (2\pi)^{-\frac{nT}{2}}|\Omega_z(\theta)|^{-\frac{1}{2}}e^{-\frac{1}{2}(z-z^\star(\theta)\otimes e_T)\Omega_z(\theta)^{-1}(z-z^\star(\theta)\otimes e_T)'}$$

- Statistical inference $\Leftrightarrow$ Reverse the conditioning, we want to compute:

$$p(\theta|\mathcal{Z}_t)$$

what can we say about the parameters knowing the data?

Bayesian inference

- The well known Bayes theorem precisely establishes how to revert a conditioning.
- Let A and B be two random variables.
- We have:

  $$p(A|B) = \frac{p(A,B)}{p(B)}$$

  and also

  $$p(B|A) = \frac{p(A,B)}{p(A)}$$

  Substituting the second equation in the first one to eliminate the joint probability:

  $$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

Bayesian inference

- Applying this result to our inference problem, we obtain:

$$p(\theta|\mathcal{Z}_T) = \frac{p(\mathcal{Z}_T|\theta)p(\theta)}{p(\mathcal{Z}_T)}$$

$$\Rightarrow p(\theta|\mathcal{Z}_T) \propto p(\mathcal{Z}_T|\theta)p(\theta)$$

- The prior beliefs are updated by the data through the likelihood.
- The posterior density, *ie* what we finally know about θ, is proportional to the likelihood times the prior density.
- $p(\mathcal{Z}_t)$, the marginal density of the sample, is only used for model comparison.

Bayesian inference

- Evaluation of the likelihood $\Rightarrow$ Kalman filters.
- Estimation of the posterior mode $\Rightarrow$ Optimization routines.
- Posterior moments and distribution $\Rightarrow$ Monte Carlo Markov Chains.

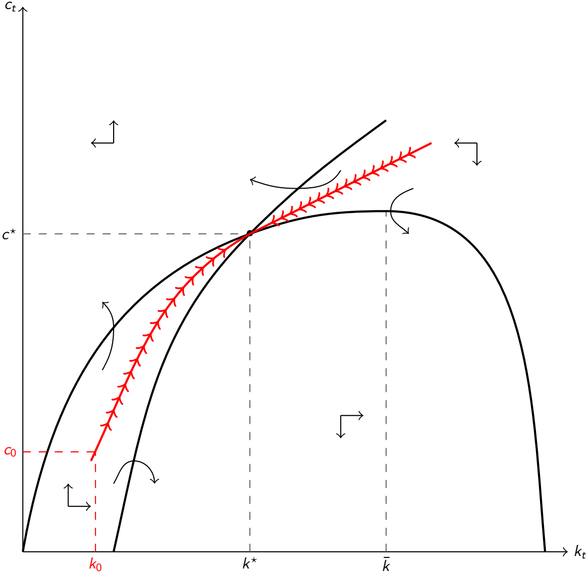
Ramsey model (Dynare code, model declaration)

```
var k y c i ;

parameters delta alpha beta ;

delta = .02;
alpha = .33;
beta  = .99;

model;
     y = k(-1)^alpha ;
     i = y - c ;
     k = i + (1-delta)*k(-1) ;
     c(1)/c =  beta*(alpha*k^(alpha-1)+1-delta) ;
end;

steady_state_model;
     k =  ( alpha/(1/beta-1+delta) )^(1/(1-alpha));
     y = k^alpha ;
     i = delta*k ;
     c = y - i ;
end;
```

Ramsey model (Dynare code, simulation)

```
kstar = ( alpha/(1/beta-1+delta) )^(1/(1-alpha));

initval;
    k = .5*kstar;
end;

endval;
    k = kstar;
end;

steady;

perfect_foresight_setup(periods=200);
perfect_foresight_solver;

plot(y);
```

Ramsey model (phase diagram)



Augmented Ramsey model

$$
\begin{aligned}
a_t &= \rho a_{t-1} + \varepsilon_t \\
y_t &= e^{a_t} k_t^{\alpha} \\
y_t &= c_t + i_t \\
k_{t+1} &= i_t + (1-\delta) k_t \\
\frac{c_{t+1}}{c_t} &= \beta \left(\alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} + 1 - \delta \right)
\end{aligned}
$$

$\Rightarrow$ What is the effect of an expected temporary rise of productivity?

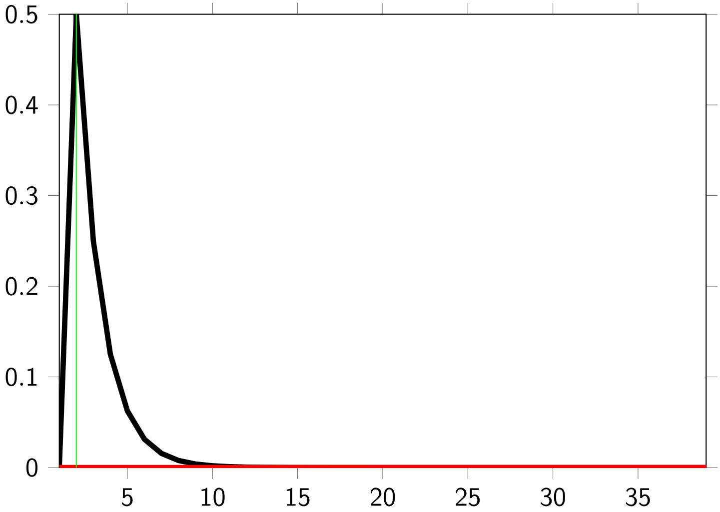
Augmented Ramsey model (Dynare code, I)

```
var k y c i a;

varexo e;

parameters delta alpha beta rho;

delta = .02;
alpha = .33;
beta  = .99;
rho   = .50;

model;
     a = rho*a(-1)+e ;
     y = exp(a)*k(-1)^alpha ;
     i = y - c ;
     k = i + (1-delta)*k(-1) ;
     c(1)/c =  beta*(alpha*exp(a(1))*k^(alpha-1)+1-delta) ;
end;
```

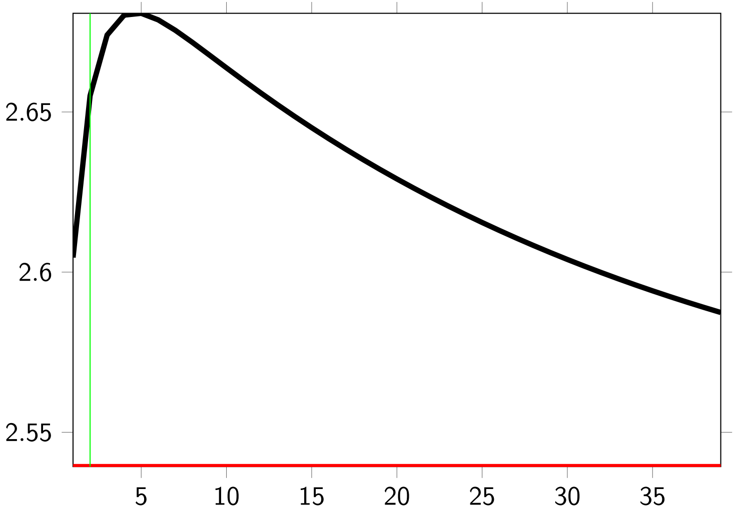
Augmented Ramsey model (Dynare code, II)

```
steady_state_model;
      a = 0;
      k = ( alpha/(1/beta-1+delta) )^(1/(1-alpha));
      y = k^alpha ;
      i = delta*k ;
      c = y - i ;
end;

steady;

shocks;
var e; periods 2; values .5;
end;

perfect_foresight_setup(periods=400);
perfect_foresight_solver;
```

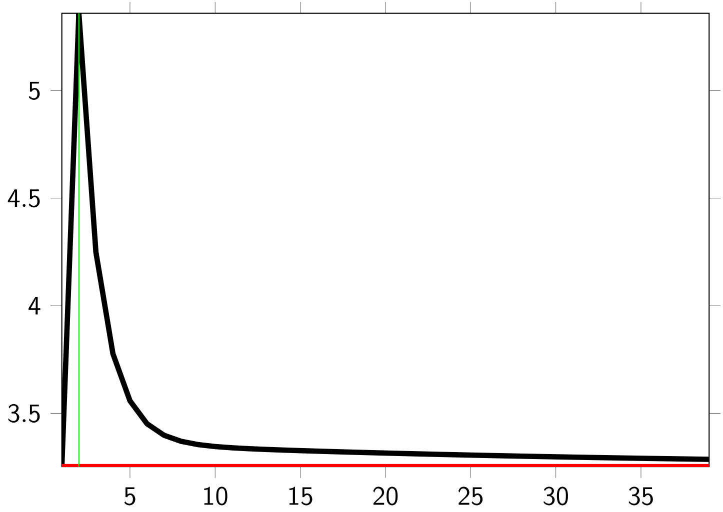
Augmented Ramsey model (Productivity path)



Augmented Ramsey model (Consumption path)



Augmented Ramsey model (Output path)

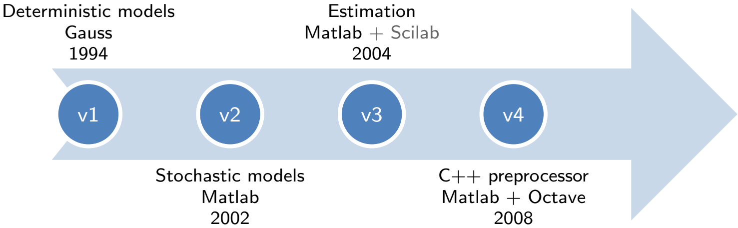


Dynare

- Dynare is a free Matlab/Octave toolbox that allows to
  - Solve RE and PF models.
  - Estimate and compare RE models.
  - Characterize the design of optimal policies.

- Dynare comes also with a preprocessor which allows the user to write models in a natural manner and translate the work to be done in Matlab/Octave/C codes (also Julia / Json output in unstable).

Dynare releases



Dynare events

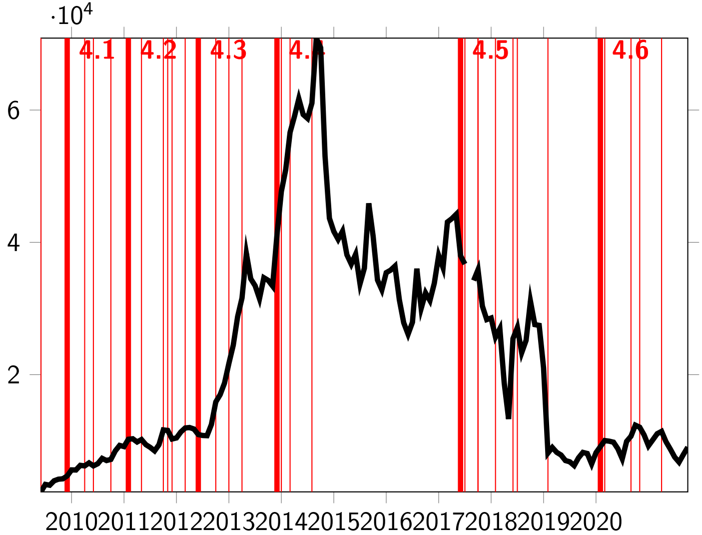
- Dynare summer school (Paris), each year since 2004.
- Dynare conference (around the world), each year since 2005.
- Workshops.
- Courses (Universities, Central banks, ...).

Dynare on the web

- www.dynare.org
  - News (releases, events, ...).
  - Documentation.
  - Downloads for Windows, OS X and Linux (source).
- Forums
  - More than 4800 users.
  - More than 60600 (true) posts.
- Wiki, Old Wiki.
- Code sources are available on our Gitlab instance.

Dynare website traffic (www.dynare.org only)

Unique visitors per month (IP)



Dynare website traffic (www.dynare.org only)

Bandwidth per month (Go, nominal)