(stochastic) Extended path

Motivations

- Nonlinearity can play an important role in macroeconomics.
- Large deviations w.r.t. the steady state or occasionally binding constraints: irreversible investment, borrowing constraint, ZLB.
- Usual local approximation techniques don't work when there are kinks.
- Deterministic models can be solved with arbitrary accuracy, unlike stochastic ones with perturbation.
- The extended path approach aims to keep the ability of deterministic methods to provide accurate account of nonlinearities.

Model to be solved

$$s_t = \mathrm{Q}(s_{t-1}, u_t) \tag{1a}$$

$$\mathrm{F}\left(y_t, x_t, s_t, \mathbb{E}_t\left[\mathscr{E}_{t+1}\right]\right) = 0 \tag{1b}$$

$$\mathrm{G}(y_t, x_{t+1}, x_t, s_t) = 0 \tag{1c}$$

$$\mathscr{E}_t = \mathrm{H}(y_t, x_t, s_t) \tag{1d}$$

s_t is a $n_s \times 1$ vector of exogenous state variables, $u_t \sim \mathrm{BB}(0, \Sigma_u)$ is a $n_u \times 1$ multivariate innovation, x_t is a $n_x \times 1$ vector of endogenous state variables, y_t is a $n_y \times 1$ vector of non predetermined variables and $\mathscr{E}_t$ is a $n_{\mathscr{E}} \times 1$ vector of auxiliary variables.

Solving perfect foresight models

- Perfect foresight models, after a shock economy returns asymptotically to equilibirum.
- For a long enough simulation, one can consider that for all practical purpose the system is back to equilibrium.
- This suggests to solve a two value boundary problem with initial conditions for some variables (backward looking) and terminal conditions for others (forward looking).
- In practice, one can use a Newton method to the equations of the model stacked over all periods of the simulation.
- The Jacobian matrix of the stacked system is very sparse and this characteristic must be used to write a practical algorithm.

Extended path approach

- Already proposed by Fair and Taylor (1983).
- The extended path approach creates a stochastic simulation as if only the shocks of the current period were random.
- Substituting (1a) in (1d), define:

$$\mathscr{E}_t = \mathscr{E}(y_t, x_t, s_{t-1}, u_t) = \mathrm{H}(y_t, x_t, \mathrm{Q}(s_{t-1}, u_t))$$

- The Euler equations (9) can then be rewritten as:

$$\mathrm{F}\big(y_t, x_t, s_t, \mathbb{E}_t\left[\mathscr{E}(y_{t+1}, x_{t+1}, s_t, u_{t+1})\right]\big) = 0$$

- The Extended path algorithm consists in replacing the previous Euler equations by:

$$\mathrm{F}\big(y_t, x_t, s_t, \mathscr{E}(y_{t+1}, x_{t+1}, s_t, 0)\big) = 0$$

Extended path algorithm

Algorithm 1 Extended path algorithm

1. $H \leftarrow$ Set the horizon of the perfect foresight (PF) model.
2. $(x^\star, y^\star) \leftarrow$ Compute steady state of the model
3. $(s_0, x_1) \leftarrow$ Choose an initial condition for the state variables
4. **for** $t = 1$ to T **do**
5. $\quad u_t \leftarrow$ Draw random shocks for the current period
6. $\quad (y_t, x_{t+1}, s_t) \leftarrow$ Solve a PF with $y_{t+H+1} = y^\star$
7. **end for**

Extended path algorithm (time t nonlinear problem)

$$
\begin{aligned}
s_t &= \mathsf{Q}(s_{t-\mathbf{1}}, u_t) \\
0 &= \mathsf{F}\big(y_t, x_t, s_t, \mathscr{E}(y_{t+\mathbf{1}}, x_{t+\mathbf{1}}, s_t, 0)\big) \\
0 &= \mathsf{G}(y_t, x_{t+\mathbf{1}}, x_t, s_t) \\
s_{t+\mathbf{1}} &= \mathsf{Q}(s_t, 0) \\
0 &= \mathsf{F}\big(y_{t+\mathbf{1}}, x_{t+\mathbf{1}}, s_{t+\mathbf{1}}, \mathscr{E}(y_{t+\mathbf{2}}, x_{t+\mathbf{2}}, s_{t+\mathbf{1}}, 0)\big) \\
0 &= \mathsf{G}(y_{t+\mathbf{1}}, x_{t+\mathbf{2}}, x_{t+\mathbf{1}}, s_{t+\mathbf{1}}) \\
&\vdots \\
s_{t+h} &= \mathsf{Q}(s_{t+h-\mathbf{1}}, 0) \\
0 &= \mathsf{F}\big(y_{t+h}, x_{t+h}, s_{t+h}, \mathscr{E}(y_{t+h+\mathbf{1}}, x_{t+h+\mathbf{1}}, s_{t+h}, 0)\big) \\
0 &= \mathsf{G}(y_{t+h}, x_{t+h+\mathbf{1}}, x_{t+h}, s_{t+h}) \\
&\vdots \\
s_{t+H} &= \mathsf{Q}(s_{t+H-\mathbf{1}}, 0) \\
0 &= \mathsf{F}\big(y_{t+H}, x_{t+H}, s_{t+H}, \mathscr{E}(y^\star, x_{t+H+\mathbf{1}}, s_{t+H}, 0)\big) \\
0 &= \mathsf{G}(y_{t+H}, x_{t+H+\mathbf{1}}, x_{t+H}, s_{t+H})
\end{aligned}
$$

Extended path algorithm (discussion)

- This approach takes full account of the *deterministic* non linearities...
- ... But neglects the Jensen inequality by setting future innovations to zero (the expectation).
- We do not solve the rational expectation model! We solve a model where the agents believe that the economy will not be perturbed in the future. They observe new realizations of the innovations in each period but do not update this belief...
- Uncertainty about the future does not matter here.
- EP > OccBin > First order perturbation

Stochastic extended path

- The strong assumption about future uncertainty can be relaxed by approximating the expected terms in the Euler equations (9):

$$\mathsf{F}\left(y_t, x_t, s_t, \mathbb{E}_t\left[\mathscr{E}_{t+1}\right]\right) = 0$$

- We assume that, at time t, agents perceive uncertainty about realizations of $u_{t+1}, \ldots, u_{t+k}$ but not about the realizations of $u_{t+\tau}$ for all $\tau > k$ (which, again, are set to zero)

- Under this assumption, the expectations are approximated using numerical integration.

Gaussian quadrature (univariate)

- Let X be a Gaussian random variable with mean zero and variance $\sigma_x^2 > 0$, and suppose that we need to evaluate $\mathbb{E}[\varphi(X)]$, where φ is a continuous function:

$$\mathbb{E}[\varphi(X)] = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-\frac{x^2}{2\sigma_x^2}} \mathrm{d}x$$

- The integral can be approximated by a finite sum, using:

$$\int_{-\infty}^{\infty} \varphi(z) e^{-z^2} \mathrm{d}x = \sum_{i=1}^{n} \omega_i \varphi(z_i) + \frac{n!\sqrt{n}}{2^n} \frac{\varphi^{(2n)}(\xi)}{(2n)!}$$

where z_i $(i = 1, \ldots, n)$ are the roots of an order n Hermite polynomial, and the weights ω_i are positive and summing up to one (the error term is zero iff φ is a polynomial of order at most $2n - 1$). $\rightarrow x_i = {z_i}/{\sigma_x\sqrt{2}}$

Gaussian quadrature (multivariate)

- Let X be a multivariate Gaussian random variable with mean zero and unit variance, and suppose that we need to evaluate

$$\mathbb{E}[\varphi(X)] = (2\pi)^{-\frac{p}{2}} \int_{\mathbb{R}^p} \varphi(\boldsymbol{x}) e^{-\frac{1}{2}\boldsymbol{x}'\boldsymbol{x}} \mathrm{d}\boldsymbol{x}$$

- Let $\{(\omega_i, z_i)\}_{i=1}^n$ be the weights and nodes of an order n univariate Gaussian quadrature.

- This integral can be approximated using a tensor grid:

$$\int_{\mathbb{R}^p} \varphi(\boldsymbol{z}) e^{-\boldsymbol{z}'\boldsymbol{z}} \mathrm{d}\boldsymbol{z} \approx \sum_{i_1,\ldots,i_p=1}^{n} \omega_{i_1} \ldots \omega_{i_p} \varphi(z_{i_1}, \ldots, z_{i_p})$$

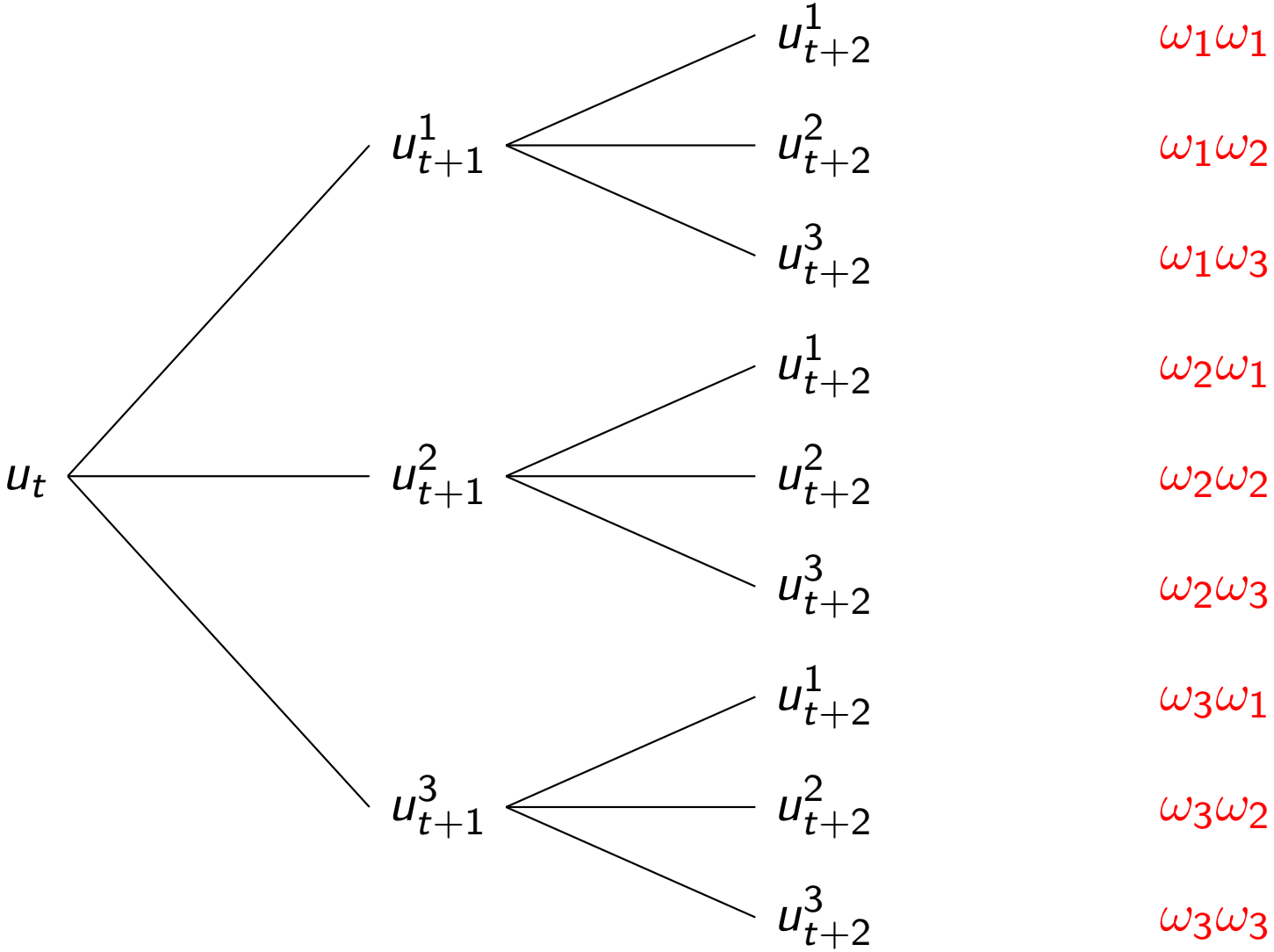
- **Curse of dimensionality 1**: The number of terms in the sum grows exponentially with the number of shocks.

Other numerical integration rules

- By default Dynare uses multivariate Gaussian quadrature to approximate the expected terms.
- To cope with **curse of dimensionality 1** we can consider other numerical integration approaches:
  - Cubature (ordre 3 or 5)
    Set `options_.ep.IntegrationAlgorithm` to `'Stroud-Cubature-3'` or `'Stroud-Cubature-5'`
  - Unscented transform
    Set `options_.ep.IntegrationAlgorithm` to `'Unscented'`
- These alternatives rely on a smaller number of nodes, which do not grow exponentially with the number of innovations.
- But there is another curse of dimensionality.

Forward histories (one shock, three nodes, order two SEP)

Curse of dimensionality 2

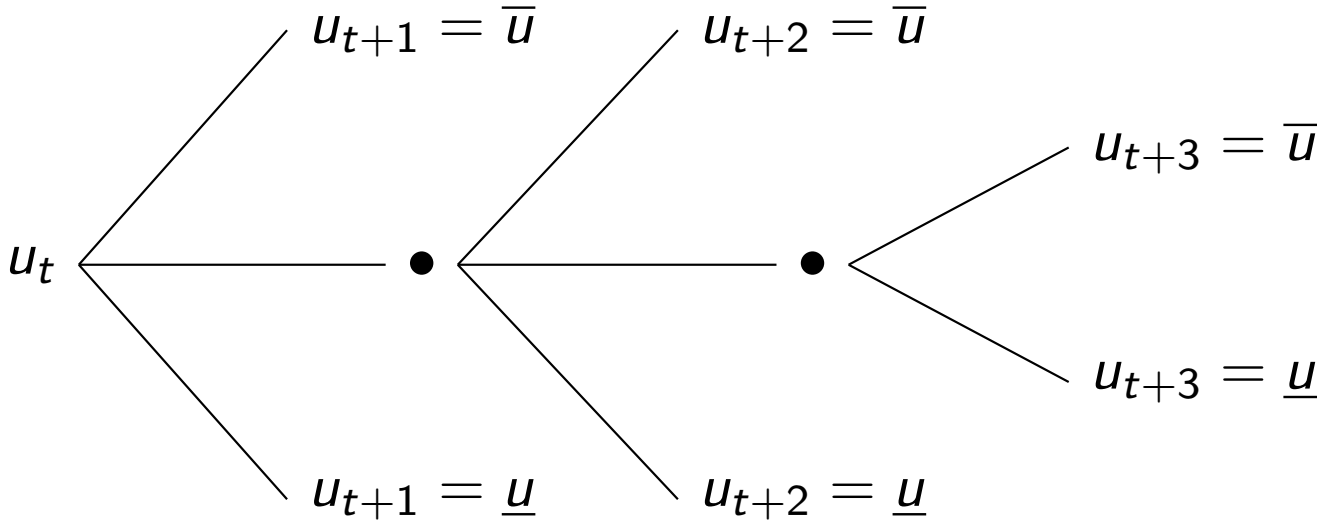


$\rightarrow$ The tree of histories grows exponentially!

Fishbone integration

- We can prune the tree of forward histories by removing low probability branches...
- Or by considering that innovations, say, at time $t+1$ and $t+2$ are unrelated variables.
- Analogy with a good at different periods in an intertemporal general equilibrium model.
- If we have n_u innovations and if agents perceive uncertainty for the next k following periods, we consider an integration problem involving $n_u \times k$ unrelated variables.
- We use a two points Cubature rule to compute the integral (unscented transform with $\kappa = 0$) $\rightarrow$ The complexity of the integration problem grows linearly with n_u or k

Fishbone history (one shock, two nodes, order three SEP)



Stochastic extended path algorithm

Algorithm 2 Stochastic Extended path algorithm

1. $H \leftarrow$ Set the horizon of the *stochastic* perfect foresight (SPF) models.
2. $(x^\star, y^\star) \leftarrow$ Compute steady state of the model.
3. $\{(\omega_i, \mu_i)\}_{i=1}^m \leftarrow$ Get weights and nodes for numerical integration
4. $(s_0, x_1) \leftarrow$ Choose an initial condition for the state variables
5. **for** $t = 1$ to T **do**
6. $\quad u_t \leftarrow$ Draw random shocks for the current period
7. $\quad (y_t, x_{t+1}, s_t) \leftarrow$ Solve a SPF model with $y_{t+H+1} = y^\star$
8. **end for**

SEP algorithm (order 1, time t nonlinear problem)

For $i = 1, \ldots, m$

$$
\begin{aligned}
s_t &= \mathsf{Q}(s_{t-\mathbf{1}}, u_t) \\
0 &= \mathsf{F}\left(y_t, x_t, s_t, \sum\nolimits_i \omega_i \mathscr{E}(y^i_{t+\mathbf{1}}, x_{t+\mathbf{1}}, s_t, \mu_i)\right) \\
0 &= \mathsf{G}(y_t, x_{t+\mathbf{1}}, x_t, s_t) \\
s^i_{t+\mathbf{1}} &= \mathsf{Q}(s_t, \mu_i) \\
0 &= \mathsf{F}(y^i_{t+\mathbf{1}}, x_{t+\mathbf{1}}, s^i_{t+\mathbf{1}}, \mathscr{E}(y^i_{t+\mathbf{2}}, x^i_{t+\mathbf{2}}, s^i_{t+\mathbf{1}}, 0)) \\
0 &= \mathsf{G}(y^i_{t+\mathbf{1}}, x^i_{t+\mathbf{2}}, x_{t+\mathbf{1}}, s^i_{t+\mathbf{1}}) \\
&\vdots \\
s^i_{t+h} &= \mathsf{Q}(s^i_{t+h-\mathbf{1}}, 0) \\
0 &= \mathsf{F}(y^i_{t+h}, x^i_{t+h}, s^i_{t+h}, \mathscr{E}(y^i_{t+h+\mathbf{1}}, x^i_{t+h+\mathbf{1}}, s^i_{t+h}, 0)) \\
0 &= \mathsf{G}(y^i_{t+h}, x^i_{t+h+\mathbf{1}}, x^i_{t+h}, s^i_{t+h}) \\
&\vdots \\
s^i_{t+H} &= \mathsf{Q}(s^i_{t+H-\mathbf{1}}, 0) \\
0 &= \mathsf{F}(y^i_{t+H}, x^i_{t+H}, s^i_{t+H}, \mathscr{E}(y^\star, x^i_{t+H+\mathbf{1}}, s^i_{t+H}, 0)) \\
0 &= \mathsf{G}(y^i_{t+H}, x^i_{t+H+\mathbf{1}}, x^i_{t+H}, s^i_{t+H})
\end{aligned}
$$

SEP algorithm (order 2, time t nonlinear problem)

For all $(i,j) \in \{1,\dots,m\}^2$

$$
\begin{aligned}
s_t &= \mathbf{Q}(s_{t-\mathbf{1}}, u_t)\\
0 &= \mathbf{F}\left(y_t, x_t, s_t, \sum\nolimits_i \omega_i \mathscr{E}(y^i_{t+\mathbf{1}}, x_{t+\mathbf{1}}, s_t, \mu_i)\right)\\
0 &= \mathbf{G}(y_t, x_{t+\mathbf{1}}, x_t, s_t)\\
s^i_{t+\mathbf{1}} &= \mathbf{Q}(s_t, \mu_i)\\
0 &= \mathbf{F}(y^i_{t+\mathbf{1}}, x_{t+\mathbf{1}}, s^i_{t+\mathbf{1}}, \sum\nolimits_j \omega_j \mathscr{E}(y^{i,j}_{t+\mathbf{2}}, x^i_{t+\mathbf{2}}, s^i_{t+\mathbf{1}}, \mu_j))\\
0 &= \mathbf{G}(y^i_{t+\mathbf{1}}, x^i_{t+\mathbf{2}}, x_{t+\mathbf{1}}, s^i_{t+\mathbf{1}})\\
s^{i,j}_{t+\mathbf{2}} &= \mathbf{Q}(s^i_{t+\mathbf{1}}, \mu_j)\\
&\vdots\\
s^{i,j}_{t+h} &= \mathbf{Q}(s^{i,j}_{t+h-\mathbf{1}}, 0)\\
0 &= \mathbf{F}(y^{i,j}_{t+h}, x^{i,j}_{t+h}, s^{i,j}_{t+h}, \mathscr{E}(y^{i,j}_{t+h+\mathbf{1}}, x^{i,j}_{t+h+\mathbf{1}}, s^{i,j}_{t+h}, 0))\\
0 &= \mathbf{G}(y^{i,j}_{t+h}, x^{i,j}_{t+h+\mathbf{1}}, x^{i,j}_{t+h}, s^{i,j}_{t+h})\\
&\vdots\\
s^{i,j}_{t+H} &= \mathbf{Q}(s^{i,j}_{t+H-\mathbf{1}}, 0)\\
0 &= \mathbf{F}(y^{i,j}_{t+H}, x^{i,j}_{t+H}, s^{i,j}_{t+H}, \mathscr{E}(y^\star, x^{i,j}_{t+H+\mathbf{1}}, s^{i,j}_{t+H}, 0))\\
0 &= \mathbf{G}(y^{i,j}_{t+H}, x^{i,j}_{t+H+\mathbf{1}}, x^{i,j}_{t+H}, s^{i,j}_{t+H})
\end{aligned}
$$

Stochastic extended path (discussion)

- The extended path approach takes full account of the deterministic nonlinearities of the model.
- It also takes into account the nonlinear effects of future shocks k-period ahead.
- It neglects the effects of uncertainty in the long run.
- The Stochastic Perfect Foresight model, that must be solved at each date, is huge.
- Curse of dimensionality with respect with the number of innovations **and** the order of approximation but *not with the number of state variables*!

Burnside (1998) model

- A representative household
- A single perishable consumption good produced by a single 'tree'.
- Household can hold equity to transfer consumption from one period to the next
- Household's intertemporal utility is given by

$$\mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^{t-\tau} \frac{c_{t+\tau}^{\theta}}{\theta} \right] \qquad \text{with } \theta \in (-\infty, 0) \cup (0, 1]$$

- Budget constraint is

$$p_t e_{t+1} + c_t = (p_t + d_t)\, e_t$$

- Dividends d_t are growing at exogenous rate x_t

$$d_t = e^{x_t} d_{t-1}$$
$$x_t = (1-\rho)\bar{x} + \rho x_{t-1} + \epsilon_t$$

Dynamics

The price/dividend ratio, $y_t = {}^{p_t}/_{d_t}$, is given by

$$y_t = \beta \mathbb{E}_t \left[e^{\theta x_{t+1}} \left(1 + y_{t+1}\right) \right]$$
$$x_t = (1-\rho)\bar{x} + \rho x_{t-1} + \epsilon_t$$

Iterating forward, y_t can be written as the current value of future dividends growth rates:

$$\begin{aligned} y_t &= \mathbb{E}_t \left[\sum_{i=1}^{\infty} \beta^{\tau} e^{\sum_{j=1}^{i} \theta x_{t+j}} \right] \\ &= \mathbb{E}_t \left[\sum_{i=1}^{\infty} \beta^{\tau} e^{\theta \sum_{j=1}^{i} \bar{x} + \rho^i \hat{x}_t + \sum_{\ell=1}^{j} \rho^{j-\ell} \epsilon_{t+\ell}} \right] \end{aligned}$$

with $\hat{x}_t = x_t - \bar{x}$.

The exact solution

Using formulas for the distribution of the log-normal random variable, Burnside (1998) shows that the closed form solution is

$$y_t = \sum_{i=1}^{\infty} \beta^i e^{a_i + b_i \hat{x}_t}$$

where

$$a_i = \theta \bar{x} i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \left(i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2} \right)$$

and

$$b_i = \frac{\theta \rho \left(1 - \rho^i\right)}{1-\rho}$$

The extended path approach

In the extended path approach, one sets future shocks to their expected value, $\mathbb{E}\left[\epsilon_{t+\ell}\right] = 0$, $\ell = 1, \ldots, \infty$. The corresponding solution is given by

$$\hat{y}_t = \sum_{i=1}^{\infty} \beta^i e^{a_i + b_i \hat{x}_t}$$

where

$$a_i = \theta \bar{x} i + \cancel{\frac{\theta^2 \sigma^2}{2(1-\rho)^2} \left(i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2} \right)}$$

and

$$b_i = \frac{\theta \rho \left(1 - \rho^i\right)}{1 - \rho}$$

Note that we abstract from the accuracy errors introduced by the numerical approximation of the integrals.

Numerical simulation

Calibration

$$\bar{x} = 0.0179$$
$$\rho = -0.139$$
$$\theta = -1.5$$
$$\beta = 0.95$$
$$\sigma = 0.0348$$

- The deterministic steady state is equal to 12.3035.
- The risky steady state, defined as the fix point in absence of shock this period:

$$\widetilde{y} = \sum_{i=1}^{\infty} \beta^i e^{\theta \bar{x} i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2}\left(i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2}\right)}$$

is equal to 12.4812.

Comparing extended path and closed-form solution

Difference between extended path approximation, $\hat{y}_t$, and closed-form solution, y_t.

- Using 800 terms to approximate the infinite summation
- Computing over 30000 periods

$$\min\left(y_t - \hat{y}_t\right) = 0.1726$$
$$\max\left(y_t - \hat{y}_t\right) = 0.1820$$

- The effect of future volatility is around 1%, $\frac{\tilde{y}-\bar{y}}{\bar{y}} = 0.0144$, and doesn't depend much on the state of the economy.

Stochastic extended path

- A k-order stochastic extended path approach computes the conditional expectation taking into accounts the shocks over the next k periods.
- The closed formula is

$$\check{y}_t = \sum_{i=1}^{\infty} \beta^i e^{a_i + b_i \hat{x}_t}$$

where

$$a_i = \theta \bar{x} i + \begin{cases} \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \left(i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2} \right) & \text{for} \quad i \leq k \\ \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \left(k - 2\rho \frac{\rho^{i-k} - \rho^i}{1-\rho} + \rho^2 \frac{\rho^{2(i-k)} - \rho^{2i}}{1-\rho^2} \right) & \text{for} \quad i > k \end{cases}$$

and

$$b_i = \frac{\theta \rho \left(1 - \rho^i\right)}{1 - \rho}$$

Quantitative evaluation

- What is the ability of the stochastic extended path approach to capture the effect of future volatility? What part of the difference between the risky steady state and deterministic steady state is captured by different values of k?

- Deterministic steady state: 12.3035

- Risky steady state: 12.4812

- The contribution of k future periods

k	Percentage
1	7.4%
2	14.3%
9	50.0%
30	90.1%
60	99.0%

Hybrid approach, I

- A very large number of periods forward (the order of stochastic extended path) is necessary to obtain an accurate figure of the effects of future volatility.
- However, even a local approximation with a Taylor expansion of low order provides better information on this effect of future volatility.
- This suggests to combine the two approaches.

Hybrid approach, II

For $i = 1, \ldots, \#\{\text{nodes}\}$:

$$
\begin{aligned}
s_t &= \mathsf{Q}(s_{t-1}, u_t) \\
0 &= \mathsf{F}\left(y_t, x_t, s_t, \sum\nolimits_i \omega_i \mathscr{E}\left(y^i_{t+1} + \frac{1}{2} g_{\sigma\sigma}, x_{t+1}, s_t, \mu_i\right)\right) \\
0 &= \mathsf{G}(y_t, x_{t+1}, x_t, s_t) \\
s^i_{t+1} &= \mathsf{Q}(s_t, \mu_i) \\
0 &= \mathsf{F}(y^i_{t+1}, x_{t+1}, s^i_{t+1}, \mathscr{E}(y^i_{t+2}, x^i_{t+2}, s^i_{t+1}, 0)) \\
0 &= \mathsf{G}(y^i_{t+1}, x^i_{t+2}, x_{t+1}, s^i_{t+1}) \\
&\vdots \\
s^i_{t+h} &= \mathsf{Q}(s^i_{t+h-1}, 0) \\
0 &= \mathsf{F}(y^i_{t+h}, x^i_{t+h}, s^i_{t+h}, \mathscr{E}(y^i_{t+h+1}, x^i_{t+h+1}, s^i_{t+h}, 0)) \\
0 &= \mathsf{G}(y^i_{t+h}, x^i_{t+h+1}, x^i_{t+h}, s^i_{t+h}) \\
&\vdots \\
s^i_{t+H} &= \mathsf{Q}(s^i_{t+H-1}, 0) \\
0 &= \mathsf{F}(y^i_{t+H}, x^i_{t+H}, s^i_{t+H}, \mathscr{E}(y^\star, x^i_{t+H+1}, s^i_{t+H}, 0)) \\
0 &= \mathsf{G}(y^i_{t+H}, x^i_{t+H+1}, x^i_{t+H}, s^i_{t+H})
\end{aligned}
$$

Hybrid approach, III

We compute the difference between the stochastic expended path approximation of order 2, the hybrid approach of order 2 and the closed-form solution, y_t. We use 800 terms to approximate the infinite summation and run simulations over 30000 periods.

	Stochastic extended path	Hybrid stochastic extended path
maximum difference	0.1607	0.0021
minimum difference	0.1513	0.0019

Irreversible investment

Consider the following RBC model with irreversible investment:

$$\max_{\{c_{t+j}, l_{t+j}, k_{t+j+1}\}_{j=0}^{\infty}} \quad \mathcal{W}_t = \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j})$$

$$\underline{s.t.}$$

$$\begin{aligned}
y_t &= c_t + i_t \\
y_t &= A_t f(k_t, l_t) \\
k_{t+1} &= i_t + (1-\delta)k_t \\
A_t &= A^\star e^{a_t} \\
a_t &= \rho a_{t-1} + \varepsilon_t \\
i_t &\geq 0
\end{aligned}$$

Further specifications

The utility function is

$$u(c_t, l_t) = \frac{\left(c_t^{\theta}(1 - l_t)^{1-\theta}\right)^{\tau}}{1 - \tau}$$

and the production function,

$$f(k_t, l_t) = \left(\alpha k_t^{\psi} + (1 - \alpha) l_t^{\psi}\right)^{\frac{1}{\psi}}$$

First order conditions

$$
\begin{aligned}
u_c(c_t, l_t) - \mu_t &= \beta \mathbb{E}_t \Big[u_c(c_{t+1}, l_{t+1}) \Big(A_{t+1} f_k(k_{t+1}, l_{t+1}) + 1 - \delta \Big) - \mu_{t+1}(1-\delta) \Big] \\
\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} &= A_t f_l(k_t, l_t) \\
c_t + k_{t+1} &= A_t f(k_t, l_t) + (1-\delta) k_t \\
0 &= \mu_t \left(k_{t+1} - (1-\delta) k_t \right)
\end{aligned}
$$

where μ_t is the Lagrange multiplier associated with the constraint on investment.

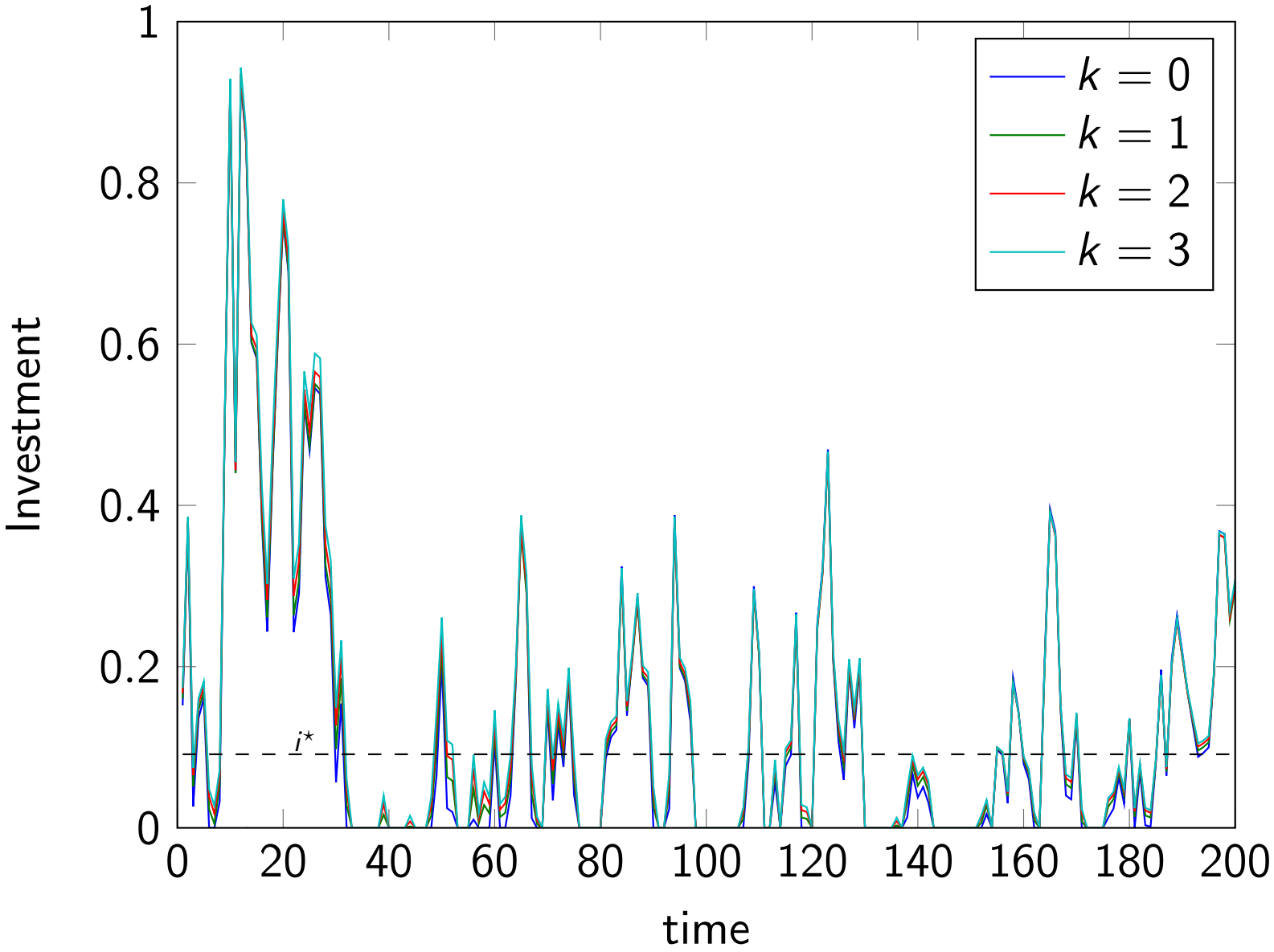
Calibration

$$\beta = 0.990$$
$$\theta = 0.357$$
$$\tau = 2.000$$
$$\alpha = 0.450$$
$$\psi = -0.500$$
$$\delta = 0.020$$
$$\rho = 0.995$$
$$A^{\star} = 1.000$$
$$\sigma = 0.100$$

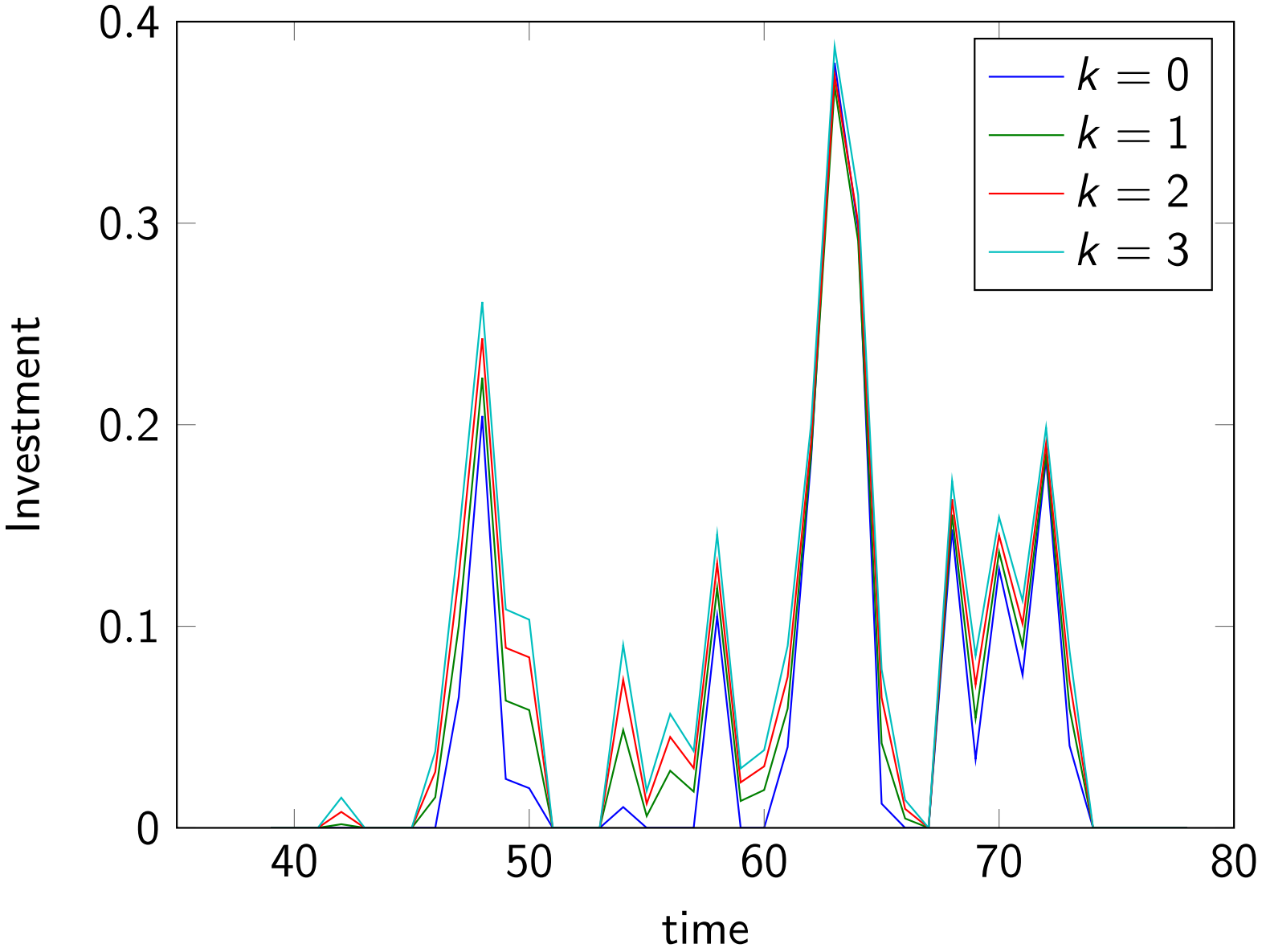
Simulation

- Order: 0, 1, 2 and 3
- Integration nodes: 3 (Gaussian quadrature)
- Number of periods for auxiliary simulations (SPF): 200

The trajectory of investment



The trajectory of investment



Conclusion and future work

- The extended path approach takes into account effects of nonlinearities.
- The stochastic extended path approach takes also partially into account nonlinear effects of future volatility.
- Possible to use an hybrid approach.
- The approach suffers from the curse of dimensionality but it can be mitigated.
- Parallelization is possible if EP is used in estimation (SMM or PF) but not in the SEP solver.
- Improvements: produce IRFs with SEP, write a mex for building the stacked SEP nonlinear problem.