Perfect foresight models

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We consider the following RBC model with irreversible investment:

$$
\begin{aligned}
\max_{\{c_{t+j},l_{t+j},k_{t+j+1}\}_{j=0}^{\infty}} \quad & \mathcal{W}_t = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) \\
\underline{s.t.} \\
y_t &= c_t + i_t \\
y_t &= A_t f(k_t, l_t) \\
k_{t+1} &= i_t + (1-\delta)k_t \\
A_t &= A^\star e^{a_t} \\
a_t &= \rho a_{t-1} + \varepsilon_t \\
i_t &\geq 0
\end{aligned}
$$

where the utility and production function are:

$$
u(c_t, l_t) = \frac{\left(c_t^\theta (1-l_t)^{1-\theta}\right)^\tau}{1-\tau}
$$

$$
f(k_t, l_t) = \left(\alpha k_t^\psi + (1-\alpha) l_t^\psi\right)^{\frac{1}{\psi}}
$$

with $0 < \theta < 1$, $\tau > 0$ and $\psi < 1$ ($1/1-\psi$ is the elasticity of substitution between capital and labour). Parameters ψ and τ allows us to control the degree of nonlinearity in the model. If $\tau \to 1$ and $\psi \to 0$, the model is almost (log)linear. The first order conditions are:

$$
\begin{aligned}
u_c(c_t, l_t) - \mu_t &= \beta \mathbb{E}_t \Big[u_c(c_{t+1}, l_{t+1}) \Big(A_{t+1} f_k(k_{t+1}, l_{t+1}) + 1 - \delta \Big) - \mu_{t+1}(1-\delta) \Big] \\
\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} &= A_t f_l(k_t, l_t) \\
c_t + k_{t+1} &= A_t f(k_t, l_t) + (1-\delta)k_t \\
0 &= \mu_t \left(k_{t+1} - (1-\delta)k_t\right)
\end{aligned}
$$

where μ is the Lagrange multiplier associated to the positivity constraint on investment.

Task 1. Write a `.mod` file for the model without the positivity constraint on investment.

Task 2. Simulate this model for different values of ψ with expected and non expected productivity shocks (initial condition is the steady state)

Task 3. Simulate transitions to the steady state for different values of ψ (initial condition is arbitrary).

Task 4. We now focus on the model with irreversible investment. A first approach is to expand the model with two regimes: in the first one $i_t > 0$ while in the second one $i_t = 0$. To this end we define choice variables in each regime: $c_{1,t}$, $l_{1,t}$, $y_{1,t}$ (because it depends on labour) in the unconstrained regime and $c_{2,t}$, $l_{2,t}$, $y_{2,t}$ in the constrained regime. In both regimes, production is defined as:

$$y_{.,t} = f(k_t, l_{.,t})$$

Consumption in the unconstrained regime (1) is defined by the Euler equation while in the constrained regime (2) consumption is equal to production. Labour supply in each regime is defined by solving:

$$\frac{u_l(c_{.,t}, l_{.,t})}{u_c(c_{.,t}, l_{.,t})} = A_t f_l(k_t, l_{.,t})$$

for $l_{.,t}$. Note that k_t, as in the definition of production, is not indexed by the regime, the capital stock is a predetermined variable so when solving for the choice variables we know in which regime capital was determined. The actual consumption, labour supply or production are determibned by the following rule:

$$x_t = \begin{cases} x_{1,t} & \text{if } y_{1,t} > c_{1,t} \text{ (investment is positive)}, \\ x_{2,t} & \text{otherwise.} \end{cases}$$

for $x = c, l,$ and y. The Euler equation, only effective in the first regime, is:

$$u_c(c_{1,t}, l_{1,t}) = \beta \mathbb{E}_t \Big[u_c(c_{t+1}, l_{t+1}) \Big(A_{t+1} f_k(k_{t+1}, l_{t+1}) + 1 - \delta \Big) - \mu_{t+1}(1-\delta) \Big]$$

note that the time t Lagrange multiplier does not appear on the left hand side since in the first regime, where investment is strictly positive, the Lagrange multiplier has to be equal to zero.
→ Write this expanded model in a `.mod` file. You will need to add an equation for the Lagrange multiplier (**hint:** $\mu_t \geq 0$ can be defined as a residual of the Euler equation, use the `max` operator).

Task 5. Find an unexpected (or expected) productivity shock such that the economy hits the zero lower bound for investment.

Task 6. Simulate time series using the extended path approach (you will find the options of the `extended_path` command in the reference manual.

Task 7. An alternative approach is to formulate our model as a Mixed Complementarity Problem (MCP). You can find a technical introduction to this type of problems in Ferris and Pang (SIAM,

1997) which provides economic applications (see section 4). Search and read in the Dynare's reference manual the documentation about the `lmmcp` option. Write a new `.mod` file that will exploit this option of the perfect foresight and extended path commands (**hint:** the `mcp` tag should be attached to the equation defining the Lagrange multiplier).

Task 8. Simulate time series using the extended path approach with option `lmmcp`.